Restricted Systems of Equations.

(Second Paper.)

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The following is a continuation of a previous paper in this Journal.† In § 3 incomplete restricted systems of defect one and two are considered, with the purpose of determining the index numbers of the residual M_1 and M_2. The results suggest the formulæ for the index numbers of a composite spread made up of an M_r and an M_s with a common M_t, where $s \overline{<} 2$, $t < s$.

In § 4 the "relative incidence numbers" of an $M_{r-k}(\gamma)$ on an $M_r(\alpha)$ are defined. They are utilized to determine a new type of index number attached to $M_r(\alpha)$, called the "residual index numbers." In case $M_r(\alpha)$ in S_n is regular or in case $M_r(\alpha)$ is an $M_{n-2}(\alpha)$, the residual index numbers can be expressed by means of the ordinary index numbers α. The new index numbers are employed also to obtain the solution of some particular cases of the more general problems of the theory.

Manifolds defined by matrices are considered in § 5. A simple proof of Salmon's formula for the order of a matrix is given; the formula for the genus of the curve defined by a matrix is derived; the third index number of the two-way defined by a matrix is determined, and all the index numbers of the manifold defined by the particular matrix with n rows and $n+1$ columns are obtained.

§ 3. *The Index Numbers of Residual Intersections and of Composite Manifolds.*

1. Given an $M_r(\alpha)$ in S_n, we have seen how the number, O_r, of points outside of $M_r(\alpha)$ and common to n spreads on $M_r(\alpha)$ can be determined in terms of the orders of the spreads and the index numbers α. This number is merely the order or first index number of the intersection residual to $M_r(\alpha)$. We are thus led to the following inquiry: Given $n-s$ spreads on $M_r(\alpha)$ in S_n, $s \overline{<} r$,

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which meet in a residual $M_s(\beta)$ which cuts $M_r(\alpha)$ in $M_{s-1}(\gamma)$, how far can the index numbers β and γ, as well as the relative index numbers $(\alpha\gamma)$ and $(\beta\gamma)$, be determined in terms of the $n-s$ given orders and the $r+1$ given index numbers α? For the particular case $s=1$ the answer to this is contained in the following theorem:

(42) *If $n-1$ spreads in S_n of orders $\lambda_1, \ldots, \lambda_{n-1}$ on $M_r(\alpha)$ meet in a residual $M_1(\beta)$ which cuts $M_r(\alpha)$ in $M_0(\gamma)$ points, then $B_0+A_{r-1}=\sigma_{n-1}$, $B_1=rC_0=rA_r$. The index numbers of a composite spread consisting of an $M_r(\alpha)$ and an $M_1(\beta)$ with $M_0(\gamma)$ common points are*

$$\alpha_0,\ \alpha_1,\ \ldots,\ \alpha_{r-2},\ \alpha_{r-1}+\beta_0,\ \alpha_r+\beta_1-(r+1)\gamma_0.$$

Let us recall that the symbol A_j has been defined as follows:

$$A_j=\alpha_j+\alpha_{j-1}\sigma_1+\ldots+\alpha_1\sigma_{j-1}+\alpha_0\sigma_j,$$

the σ's being the elementary symmetric polynomials in the given orders λ. The B_j and C_j are similarly defined for the index numbers β and γ. The formulæ given above determine β_0, β_1, and $\gamma_0=(\alpha\gamma)_0=(\beta\gamma)_0$ in terms of the orders λ and the index numbers α.

According to [(24), "R. S.," I] the theorem is true for $r=1$. Let us assume it to be true for all values of the dimension up to the given r. An n-th spread of order λ on $M_r(\alpha)$ determines O_r points outside of $M_r(\alpha)$. This number O_r is $\lambda\beta_0-\gamma_0$ or, from [(7), "R. S.," I], is $\lambda\sigma_{n-1}-(A_r+\lambda A_{r-1})$. Equating coefficients of the arbitrary order λ, we find that $B_0+A_{r-1}=\sigma_{n-1}$ and $C_0=A_r$. Again, a spread of order μ on $M_1(\beta)$ cuts $M_r(\alpha)$ in $M_{r-1}(\alpha')$, which meets $M_1(\beta)$ in $M_0(\gamma)$ points. Then $M_{r-1}(\alpha')$ and $M_1(\beta)$ together constitute a composite manifold $M_{r-1}(\varepsilon)$ which is a complete intersection, and $O_{r-1}=0$ $=\mu\sigma_{n-1}-E_{r-1}-\mu E_{r-2}$. But according to (42), for the dimension $r-1$, E_{r-2} $=A'_{r-2}+B_0$, and $E_{r-1}=A'_{r-1}+B_1-rC_0$. Furthermore, according to [(16), "R. S.," I], $A'_{r-1}+\mu A'_{r-2}=\mu A_{r-1}$, whence $0=\mu\sigma_{n-1}-\mu A_{r-1}-\mu B_0-B_1+rC_0$. Equating coefficients of μ, we find again that $\sigma_{n-1}=A_{r-1}+B_0$, and further that $B_1=rC_0$, which proves the first part of (42).

The $M_r(\alpha)$ and $M_1(\beta)$ constitute a complete $M_r(\delta)$ for which $D_r=0$. We know that $\delta_0=\alpha_0, \ldots, \delta_{r-2}, \alpha_{r-2}, \delta_{r-1}=\alpha_{r-1}+\beta_0$; let us assume that $\delta_r=\alpha_r+\beta_1-x$. Since $D_r=A_r+B_1-x=0$, we see that $x=A_r+B_1=C_0+rC_0=(r+1)\gamma_0$, which for this case at least proves the second part of (42).

A proof which applies generally can be formulated by the aid of the following lemma:

(43) *If $n-r+k$ spreads of orders $\lambda_1, \ldots, \lambda_{n-r+k}$ on $M_r(\alpha)$ in S_n meet in a residual $M_{r-k}(\beta)$ which cuts $M_r(\alpha)$ in an $M_{r-k-1}(\gamma)$, the relative index numbers of $M_{r-k-1}(\gamma)$ as to $M_{r-k}(\beta)$ are $(\beta\gamma)_i=A_{i+k+1}$, $(i=0, 1, \ldots, r-k-1)$.*

Let $r-k$ further spreads of orders $\tau(\mu_1, \ldots, \mu_{r-k})$* on $M_r(\alpha)$ meet in O_r points outside $M_r(\alpha)$, where

$$O_r = \sigma_{n-r+k}\tau_{r-k} - (A_r + A_{r-1}\tau_1 + \ldots + A_{k+1}\tau_{r-k-1} + A_k\tau_{r-k}).$$

Since the spreads $\tau(\mu)$ cut $M_{r-k}(\beta)$ in the same number of points outside $M_{r-k-1}(\gamma)$,

$$O_r = \beta_0\tau_{r-k} - \{(\beta\gamma)_{r-k-1}(\beta\gamma)_{r-k-2}\tau_1 + \ldots + (\beta\gamma)_1\tau_{r-k-2} + (\beta\gamma)_0\tau_{r-k-1}\}.$$

Noting that $\beta_0 = \sigma_{n-r+k} - A_k$, the lemma is proved by equating coefficients of τ.

We need also the further fact:

(44) *The theorem (42) applies also to the case where α, β, γ are the relative index numbers of $M_r(\alpha)$, $M_1(\beta)$, and $M_0(\gamma)$ with respect to a manifold M_n containing them, provided σ_{n-1} is replaced by $m\sigma_{n-1}$, where m is the order of M_n.*

The proof of (44) parallels that of (42). Suppose, then, that the second part of (42) and of (44) also has been established for values of the dimension up to the value r. Let spreads $\sigma(\lambda_1, \ldots, \lambda_{n-1})$ on $M_r(\alpha)$ and $M_1(\beta)$ meet again in $M_r(\alpha')$, which contains $M_1(\beta)$ and which meets $M_r(\alpha)$ in $M_{r-1}(\alpha'')$. According to (43), $(\alpha'\alpha'')_i = A_{i+1}$. According to [(38), "R. S.," I], the relative index numbers of $M_1(\beta)$, on the composite spread $M_r(\varkappa)$ made up of $M_r(\alpha)$ and $M_r(\alpha')$, are $(\varkappa\beta)_0 = \beta_0$ and $(\varkappa\beta)_1 = \beta_1 + \sigma_1\beta_0$. These relative index numbers can be determined by cutting $M_r(\varkappa)$ by r spreads $\rho(\nu)$ on $M_1(\beta)$ from the equation $O_1 = \rho_r(\alpha_0 + \alpha'_0) - (\varkappa\beta)_0\rho_1 - (\varkappa\beta)_1$. The O_1 points are made up of $\rho_r\alpha_0 - \gamma_0$ points on $M_r(\alpha)$, and of $\rho_r\alpha'_0 - (\alpha'\beta)_0\rho_1 - (\alpha'\beta)_1$ points on $M_r(\alpha')$. Since $\beta_0 = (\varkappa\beta)_0 = (\alpha'\beta)_0$, we find that $(\alpha'\beta)_1 = \beta_1 + \beta_0\sigma_1 - \gamma_0$. For the dimension $r-1$, we have assumed that the relative index numbers of $M_{r-1}(\alpha'')$ and $M_1(\beta)$ on $M_r(\alpha')$ are $(\alpha'\alpha'')_{r-1} + (\alpha'\beta)_1 - r\gamma_0$, $(\alpha'\alpha'')_{r-2} + (\alpha'\beta)_0$, $(\alpha'\alpha'')_{r-3}, \ldots$, whence spreads $\tau(\mu_1, \ldots, \mu_r)$ on the two meet $M_r(\alpha')$ in

$$\begin{aligned} O'_{r-1} = \tau_r\alpha'_0 &- [(\alpha'\alpha'')_{r-1} + (\alpha'\beta)_1 - r\gamma_0] \\ &- [(\alpha'\alpha'')_{r-2} + (\alpha'\beta)_0]\tau_1 - (\alpha'\alpha'')_{r-3}\tau_2 - \ldots - (\alpha'\alpha'')_0\tau_{r-1} \end{aligned}$$

further points. This can be written as

$$\begin{aligned} O'_{r-1} = \sigma_{n-r}\tau_r - \tau_r\alpha_0 &- [A_r + B_1 - (r+1)\gamma_0] \\ &- [A_{r-1} + B_0]\tau_1 - A_{r-2}\tau_2 - \ldots - A_1\tau_{r-1}. \end{aligned}$$

Since this is also the number O_r of points outside of $M_r(\alpha)$ and $M_1(\beta)$ on the $\sigma(\lambda)$ and $\tau(\mu)$ spreads containing them, we see that the last index number of the composite spread must be $\alpha_r + \beta_1 - (r+1)\gamma_0$, which completes the proof of (42). A similar argument completes the proof of (44).

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* This notation indicates that $\tau_1, \tau_2, \ldots$ are the elementary symmetric functions of the given orders.

Further theorems of the following type:

(45) *The index numbers of a composite curve composed of three curves* $M_1(\alpha)$, $M_1(\beta)$, $M_1(\gamma)$, *with respectively* ζ_0, η_0, ϑ_0 *points common to two and with points common to the three of which* δ_0 *have non-coplanar tangents and* δ_0' *have coplanar tangents, are* $\alpha_0+\beta_0+\gamma_0$ *and* $\alpha_1+\beta_1+\gamma_1-2\zeta_0-2\eta_0-2\vartheta_0-4\delta_0-6\delta_0'$,

might be given; but the number of particular cases increases rapidly with the dimension.

2. Let us next consider the case where $n-2$ spreads $\sigma(\lambda)$ on $M_2(\alpha)$ in S_n meet again in $M_2(\beta)$, which cuts $M_2(\alpha)$ in $M_1(\gamma)$. By taking a section and applying (42), we find that $B_0+A_0=\sigma_{n-2}$ and $B_1=C_0=A_1$. By applying the lemma (43), we find that $(\beta\gamma)_1=A_2$, and from the symmetry that $(\alpha\gamma)_1=B_2$. A further spread of order μ on $M_2(\alpha)$ meets $M_2(\beta)$ in $M_1(\gamma)$ and in $M_1(\delta)$, which has ε_0 points in common with $M_1(\gamma)$. Again applying (42), we find that $D_1+\mu D_0=2\varepsilon_0$. Considering $M_1(\gamma)$ and $M_1(\delta)$ on the one hand as a complete intersection of $M_2(\beta)$, and on the other as a composite curve, we have, according to [(17), "R. S.," I] and (42), the index numbers $\mu\beta_0=\gamma_0+\delta_0$, $\mu\beta_1-\mu^2\beta_0=\gamma_1+\delta_1-2\varepsilon_0$. By adding $(\sigma_1+\mu)$ times the first to the second, we get $\mu B_1=C_1+\mu C_0+D_1+\mu D_0-2\varepsilon_0$, whence $C_1=0$. Collecting the above equations, we have

$$(46)\quad B_0+A_0=\sigma_{n-2},\quad B_1=C_0=A_1,\quad B_2=(\alpha\gamma)_1,\quad A_2=(\beta\gamma)_1,\quad C_1=0.$$

From these we derive

$$B_0+A_0=\sigma_{n-2},\quad B_1+A_1-2C_0=0,\quad B_2+A_2-2C_1-[(\beta\gamma)_1+(\alpha\gamma)_1]=0.$$

The composite spread $M_2(\alpha)$, $M_2(\beta)$ is regular, and these equations show, according to [(9), "R. S.," I], that the index numbers of the composite spread are

$$(47)\quad \alpha_0+\beta_0,\ \alpha_1+\beta_1-2\gamma_0,\quad \alpha_2+\beta_2-2\gamma_1-[(\alpha\gamma)_1+(\beta\gamma)_1].$$

(48) *If* $n-2$ *spreads on* $M_2(\alpha)$ *in* S_n *meet again in* $M_2(\beta)$, *which cuts* $M_2(\alpha)$ *in* $M_1(\gamma)$, *the index numbers* β, γ, $(\alpha\gamma)$, $(\beta\gamma)$, *with the exception of either* β_2 *or* $(\alpha\gamma)_1$, *are determined in* (46). *The index numbers of the composite spread* $M_2(\alpha)$, $M_2(\beta)$ *with common* $M_1(\gamma)$ *are given in* (47).

We have obtained (47) in the particular case of a composite regular intersection. They are evaluated for the general case below. For the present they may be checked by thinking of the composite spread and of its parts as lying in an S_{n+1} containing S_n. According to [(13) and (34), "R. S.," I] the same formulæ hold, as of course they should.

Next let us consider the residual intersection $M_2(\beta)$ when $M_r(\alpha)$ is an $M_3(\alpha)$. Using the same method as above, we find that $B_0+A_1=\sigma_{n-2}$, $B_1=2C_0=2A_2$, $(\beta\gamma)_1=A_3$, $\varepsilon_0=A_3+\mu A_2$. But in this case $D_1+\mu D_0=3\varepsilon_0$, so that $C_1+A_3=0$. If a spread of order ν on $M_2(\beta)$ cuts $M_3(\alpha)$ in $M_2(\eta)$, where $\eta_0=\nu\alpha_0$, $\eta_1=\nu\alpha_1-\nu^2\alpha_0$, $\eta_2=\nu\alpha_2-\nu^2\alpha_1+\nu^3\alpha_0$, and where, according to [(39), "R. S.," I], $(\eta\gamma)_1=(\alpha\gamma)_1+\nu\gamma_0$, then $M_2(\eta)$ and $M_2(\beta)$ constitute an $M_2(\varkappa)$, which is a complete manifold. Therefore $K_2+\nu K_1=0$, where

$$\begin{aligned}\varkappa_0&=\nu\alpha_0 &&+\beta_0\\ \varkappa_1&=\nu\alpha_1-\nu^2\alpha_0 &&+\beta_1-2\gamma_0\\ \varkappa_2&=\nu\alpha_2-\nu^2\alpha_1+\nu^3\alpha_0&&+\beta_2-2\gamma_1-[(\alpha\gamma)_1+\nu\gamma_0+(\beta\gamma)_1].\end{aligned}$$

Thus we find that $B_2-2C_1-[(\beta\gamma)_1+(\alpha\gamma)_1]=0$ or

$$(49)\qquad \begin{cases}B_0+A_1=\sigma_{n-2}, & B_1=2C_0=2A_2,\\ C_1=-A_3=-(\beta\gamma)_1, & B_2+A_3=(\alpha\gamma)_1.\end{cases}$$

These equations lead to

$$B_1+A_1=\sigma_{n-2},\quad B_1+A_2-3C_0=0,\quad B_2+A_3-3C_1-[(\alpha\gamma)_1+3(\beta\gamma)_1]=0,$$

which proves that the index numbers of the composite spread $M_3(\alpha)$, $M_2(\beta)$ are

$$(50)\qquad \alpha_0,\ \alpha_1+\beta_0,\ \alpha_2+\beta_1-3\gamma_0,\ \alpha_3+\beta_2-3\gamma_1-[(\alpha\gamma)_1+3(\beta\gamma)_1].$$

The above argument by which the result for $M_3(\alpha)$ is gotten from that for $M_2(\alpha)$ can be applied similarly to obtain analogous formulæ for an $M_r(\alpha)$ from those for an $M_{r-1}(\alpha)$. Thus a readily formulated deduction leads to the theorem:

(52) *In S_n, $n-2$ spreads on $M_r(\alpha)$ meet in a residual $M_2(\beta)$ which cuts $M_r(\alpha)$ in $M_1(\gamma)$. The index numbers β, γ, $(\alpha\gamma)$, and $(\beta\gamma)$, with the exception of either β_2 or $(\alpha\gamma)_1$, are determined in terms of α and the given orders by*

$$\begin{aligned}B_0+A_{r-2}&=\sigma_{n-2}, & B_1&=(r-1)C_0=(r-1)A_{r-1},\\ -C_1&=(r-2)A_r=(r-2)(\beta\gamma), & B_2&+\binom{r-1}{2}A_r=(\alpha\gamma)_1.\end{aligned}$$

The index numbers of the composite spread $M_r(\alpha)$, $M_2(\beta)$ with common $M_1(\gamma)$ are

$$\alpha_0,\ \alpha_1,\ \ldots,\alpha_{r-3},\ \alpha_{r-2}+\beta_0,\ \alpha_{r-1}+\beta_1-2\gamma_0,\ \alpha_r+\beta_2-2\gamma_1-[(\alpha\gamma)_1+\binom{r}{2}(\beta\gamma)_1].$$

Only in the particular case $r=n-2$ is the determination of β_2 made later (see the end of § 4). In fact, it seems probable that the orders and the index numbers α do not constitute, in general, sufficient data to determine β_2. Further index numbers of $M_r(\alpha)$ can be defined in terms of which β_2 can be expressed, but this is not done in this paper.

Let us obtain directly the index numbers of a composite spread $M_2(\alpha)$, $M_2(\beta)$ with common $M_1(\vartheta)$. Spreads $\sigma(\lambda)$ on the two meet in a residual $M_2(\gamma)$ which cuts $M_2(\alpha)$ in $M_1(\eta)$, and $M_2(\beta)$ in $M_1(\zeta)$, and $M_1(\vartheta)$ in $M_0(\delta)$. From the first equation of (46) we get

1°. $\alpha_0+\beta_0+\gamma_0=\sigma_{n-2}$.

From each of the other equations of (46) we deduce three equations according as $M_2(\alpha)$, $M_2(\beta)$, or $M_2(\gamma)$ is looked upon as the residual manifold. These equations are

$$2^\circ.\quad \begin{cases}\zeta_0+\eta_0+\vartheta_0=\zeta_0+\alpha_0\sigma_1+\alpha_1=(\beta_0+\gamma_0)\sigma_1+\beta_1+\gamma_1-\zeta_0,\\ \zeta_0+\eta_0+\vartheta_0=\eta_0+\beta_0\sigma_1+\beta_1=(\gamma_0+\alpha_0)\sigma_1+\gamma_1+\alpha_1-\eta_0,\\ \zeta_0+\eta_0+\vartheta_0=\vartheta_0+\gamma_0\sigma_1+\gamma_1=(\alpha_0+\beta_0)\sigma_1+\alpha_1+\beta_1-\vartheta_0,\end{cases}$$

$$3^\circ.\quad \begin{Bmatrix}\sigma_1(\eta_0+\vartheta_0)+\eta_1+\vartheta_1-2\delta_0=0,\\ \sigma_1(\vartheta_0+\zeta_0)+\vartheta_1+\zeta_1-2\delta_0=0,\\ \sigma_1(\zeta_0+\eta_0)=\zeta_1+\eta_1-2\delta_0=0,\end{Bmatrix}\ \text{or}\ \begin{cases}\sigma_1\zeta_0+\zeta_1=\delta_0,\\ \sigma_1\eta_0+\eta_1=\delta_0,\\ \sigma_1\vartheta_0+\vartheta_1=\delta_0,\end{cases}$$

$$4^\circ.\quad \begin{cases}\sigma_2\alpha_0+\sigma_1\alpha_1+\alpha_2=(\gamma\eta)_1+(\beta\vartheta)_1,\\ \sigma_2\beta_0+\sigma_1\beta_1+\beta_2=(\alpha\vartheta)_1+(\gamma\zeta)_1,\\ \sigma_2\gamma_0+\sigma_1\gamma_1+\gamma_2=(\beta\zeta)_1+(\alpha\eta)_1,\end{cases}$$

$$5^\circ.\quad \begin{cases}\sigma_2(\beta_0+\gamma_0)+\sigma_1(\beta_1+\gamma_1-2\zeta_0)+x_{\beta\gamma}=(\alpha\eta)_1+(\alpha\vartheta)_1-2\delta_0,\\ \sigma_2(\gamma_0+\alpha_0)+\sigma_1(\gamma_1+\alpha_1-2\eta_0)+x_{\gamma\alpha}=(\beta\vartheta)_1+(\beta\zeta)_1-2\delta_0,\\ \sigma_2(\alpha_0+\beta_0)+\sigma_1(\alpha_1+\beta_1-2\vartheta_0)+x_{\alpha\beta}=(\gamma\zeta)_1+(\gamma\eta)_1-2\delta_0,\end{cases}$$

where $x_{\beta\gamma}$, etc., are the unknown third index numbers of the composite spread, $M_2(\beta)$, $M_2(\gamma)$, etc. From 1° γ_0 is obtained. Equations 2° reduce to three which determine γ_1, η_0, ζ_0 in terms of α_0, α_1, β_0, β_1, ϑ_0, γ_0. Then from 3° δ_0, ζ_1, η_1 are obtained in terms of ζ_0, η_0, ϑ_0. From 4° $(\gamma\eta)_1+(\gamma\zeta)_1$ is determined in terms of α_i, β_i, $(\alpha\vartheta)_1$, and $(\beta\vartheta)_1$; and, finally, from 5° we get $x_{\alpha\beta}$, which turns out to be $\alpha_2+\beta_2-2\vartheta_1-[(\alpha\vartheta)_1+(\beta\vartheta)_1]$.

By adding 2° we find that

$$\sigma_1(\alpha_0+\beta_0+\gamma_0)+\alpha_1+\beta_1+\gamma_1-2\zeta_0-2\eta_0-2\vartheta_2=0;$$

and by adding 3° and 4°, that

$$6^\circ.\quad \sigma_2(\alpha_0+\beta_0+\gamma_0)+\sigma_1(\alpha_1+\beta_1+\gamma_1-2\zeta_0-2\eta_0-2\vartheta_0) \\ +[\alpha_2+\beta_2+\gamma_2-2\zeta_1-2\eta_1-2\vartheta_1-\{(\alpha\eta)_1+(\alpha\vartheta)_1+(\beta\vartheta)_1 \\ +(\beta\zeta)_1+(\gamma\zeta)_1+(\gamma\eta)_1\}+6\delta_0]=0.$$

Since $M_2(\alpha)$, $M_2(\beta)$, $M_2(\gamma)$ constitute a regular intersection, this shows that

(52) *The index numbers of $M_2(\alpha)$, $M_2(\beta)$, $M_2(\gamma)$ with curves $M_1(\zeta)$, $M_1(\eta)$, $M_1(\vartheta)$ common to two respectively, and points $M_0(\delta)$ common to the three, are the coefficients of σ in 6°.*

We get the same result by using the above formula for $x_{\alpha\beta}$, taking $M_2(\alpha)$, $M_2(\beta)$ as a manifold $M_2(\varkappa)$, and $M_2(\gamma)$ as the residual spread meeting $M_2(\varkappa)$ in $M_1(\lambda)=M_1(\zeta), M_1(\eta)$. For $\gamma_2=\gamma_2$, $\varkappa_2=\alpha_2+\beta_2-2\vartheta_1-[(\alpha\vartheta_1)+(\beta\vartheta)_1]$, $\lambda_1=\zeta_1+\eta_1-2\delta_0$, $(\gamma\lambda)_1=(\gamma\eta)_1+(\gamma\zeta)_1-2\delta_0$, and $(\varkappa\lambda)_1=(\alpha\eta)_1+(\beta\zeta)_1$. Hence,

$$\varkappa_2+\gamma_2-2\lambda_1-[(\varkappa\lambda)_1+(\gamma\lambda)_1]=\alpha_2+\beta_2+\gamma_2-2(\zeta_1+\eta_1+\vartheta_1)$$
$$-[(\alpha\eta)_1+(\alpha\vartheta)_1+(\beta\vartheta)_1+(\beta\zeta)_1+(\gamma\zeta)_1+(\gamma\eta)_1]+6\delta_0.$$

The theorems above can be generalized, as in (44), to apply to the relative index numbers of the $M_2(\beta)$ residual to an $M_r(\alpha)$ on an M_n.

§ 4. *Residual Index Numbers.*

1. Given an $M_{r-k}(\varepsilon)$ on an $M_r(\alpha)$ in S_n; then $n-k-1$ spreads $\sigma(\lambda)$ on $M_r(\alpha)$ meet in a residual $M_{k+1}(\beta)$ which has an $M_k(\gamma)$ in common with $M_r(\alpha)$. This $M_k(\gamma)$ meets $M_{r-k}(\varepsilon)$ in I_{r-k} points, and we define the *relative incidence number*, $[\alpha\varepsilon]_{r-k}$, *of* $M_{r-k}(\varepsilon)$ *as to* $M_r(\alpha)$ by means of the equation

$$(53)\qquad I_{r-k}=\sigma_{r-k}[\alpha\varepsilon]_0+\sigma_{r-k-1}[\alpha\varepsilon]_1+\sigma_{r-k-2}[\alpha\varepsilon]_2$$
$$+\dots+\sigma_1[\alpha\varepsilon]_{r-k-1}+[\alpha\varepsilon]_{r-k},$$

in terms of the orders λ, the number I_{r-k}, and the earlier incidence numbers, $[\alpha\varepsilon]_{r-k-1},\dots,[\alpha\varepsilon]_0$, which are similarly defined for successive sections, in particular $[\alpha\varepsilon]_0$ being ε_0. We shall prove that

(54) *The relative incidence numbers are independent of the orders of the spreads used to define them, and they depend on the underlying dimension S_n just as do the ordinary index numbers.*

For if λ_1 increase by one, its spread by an S_{n-1} which cuts $M_r(\alpha)$ in $M_{r-1}(\alpha)$, then the $n-k-2$ spreads $\sigma'(\lambda_2,\dots,\lambda_{n-k-1})$ on $M_{r-1}(\alpha)$ meet in a residual $M'_{k+1}(\beta)$ in S_{n-1} which meets $M_{r-1}(\alpha)$ in $M'_k(\gamma)$. This $M'_k(\gamma)$ meets $M_{r-k-1}(\varepsilon)$ in

$$I'_{r-k-1}=\sigma'_{r-k-1}[\alpha\varepsilon]_0+\sigma'_{r-k-2}[\alpha\varepsilon]_1+\dots+\sigma'_1[\alpha\varepsilon]_{r-k-2}+[\alpha\varepsilon]_{r-k-1}$$

points. Hence, I_{r-k} is increased by I'_{r-k-1}, which is precisely the increase in I_{r-k} of (54) due alone to the change in λ_1; *i. e.*, $[\alpha\varepsilon]_{r-k}$ is unaltered, and it is independent of the order λ_1. If, however, $M_r(\alpha)$ be supposed to lie in an S_{n+1} containing S_n, we must use $\sigma(1,\lambda_1,\dots,\lambda_{n-k-1})$ for the same I_{r-k}. If $[\alpha\varepsilon]'_i+[\alpha\varepsilon]'_{i-1}=[\alpha\varepsilon]_i$ be assumed true for $i=1,\dots,r-k-1$, as it is for $i=0$, then $[\alpha\varepsilon]'_{r-k}+[\alpha\varepsilon]'_{r-k-1}=[\alpha\varepsilon]_{r-k}$. Here the $[\alpha\varepsilon]'_i$ refer to the relative incidence numbers in S_{n+1}. Comparing these relations with [(13) and (14), "R. S.," I], we see that the dependence of the relative incidence numbers upon the underlying dimension is the same as that of the ordinary index numbers. The generalization of this result analogous to [(17) and (36), "R. S.," I] is:

(55) *If* $[\alpha\varepsilon]_i$ *are the relative incidence numbers of* $M_{r-k}(\varepsilon)$ *as to* $M_r(\alpha)$, *the relative incidence numbers* $[\alpha\varepsilon]'_i$ *of* $M_{r-k-1}(\varepsilon)$ *as to* $M_{r-1}(\alpha)$, *the meets of* $M_{r-k}(\varepsilon)$ *and of* $M_r(\alpha)$ *with a spread of order* q, *are given by*

$$[\alpha\varepsilon]'_i=q[\alpha\varepsilon]_i-q^2[\alpha\varepsilon]_{i-1}+q^3[\alpha\varepsilon]_{i-2}-\ldots+(-1)^i q^{i+1}[\alpha\varepsilon]_0, \quad (i=0,\ldots,r-k-1).$$

We might expect the relative incidence numbers to behave like the ordinary index numbers, since the latter are a special case of the former. For if $M_{r-k}(\varepsilon)$ coincides with $M_r(\alpha)$, k is zero and I_r becomes the $C_0=A_r$ of (42). Comparing A_r with the right-hand member of (53) and noting that, in this case, $[\alpha\varepsilon]_0=\varepsilon_0=\alpha_0$, we have $[\alpha\varepsilon]_i=\alpha_i$.

(56) *The relative incidence numbers of* $M_r(\alpha)$ *as to* $M_r(\alpha)$ *itself are the ordinary index numbers of* $M_r(\alpha)$.

We see from the definition that the relative incidence numbers of $M_a(\varepsilon)$ and $M_{a'}(\varepsilon')$ on $M_r(\alpha)$ are the sums of the respective relative incidence numbers of $M_a(\varepsilon)$ and of $M_{a'}(\varepsilon')$ if $a=a'$; otherwise they are the same as those of the manifold of greater dimension.

(57) *If* $M_{r-k}(\varepsilon)$ *is the regular intersection of* $M_r(\alpha)$ *and an* M_{n-k} *of order* q, *then* $[\alpha\varepsilon]_i=q\alpha_i$, $(i=0, 1, \ldots, r-k)$.

Since $[\alpha\varepsilon]_0=\varepsilon_0=q\alpha_0$, let us assume that $[\alpha\varepsilon]_i=q\alpha_i$ for $i=0,\ldots,r-k-1$ and determine $[\alpha\varepsilon]_{r-k}$. Let $n-k-1$ spreads $\sigma(\lambda)$ on $M_r(\alpha)$ meet again in $M_{k+1}(\beta)$, which cuts $M_r(\alpha)$ in $M_k(\gamma)$. If $M_k(\gamma)$ meets $M_{r-k}(\varepsilon)$ in I_{r-k} points, these points are the meets of $M_k(\gamma)$ and M_{n-k}, and

$$I_{r-k}=q\gamma_0=\sigma_{r-k}[\alpha\varepsilon]_0+\sigma_{r-k-1}[\alpha\varepsilon]_1+\ldots+\sigma_1[\alpha\varepsilon]_{r-k-1}+[\alpha\varepsilon]_{r-k}.$$

By applying (42) to a proper section, we find that

$$\gamma_0=\sigma_{r-k}\alpha_0+\sigma_{r-k-1}\alpha_1+\ldots+\sigma_1\alpha_{r-k-1}+\alpha_{r-k}.$$

Multiplying by q and subtracting, we have $[\alpha\varepsilon]_{r-k}=q\alpha_{r-k}$.

(58) *If* $M_{r-k}(\varepsilon)$ *on* $M_r(\alpha)$ *be cut regularly by an* M_{n-l} *of order* q *in an* $M_{r-k-l}(\varepsilon')$, *then the relative incidence numbers of* $M_{r-k-l}(\varepsilon')$ *on* $M_r(\alpha)$ *are* $[\alpha\varepsilon']_i=q[\alpha\varepsilon]_i$, $(i=0, 1, \ldots, r-k-l)$.

This being true for $[\alpha\varepsilon']_0$, let us assume it true for $i=1,\ldots,r-k-l-1$. If $n-k-l-1$ spreads $\sigma(\lambda)$ on $M_r(\alpha)$ meet in a residual $M_{k+l+1}(\beta)$ which cuts $M_r(\alpha)$ in $M_{k+l}(\gamma)$, the $M_{k+l}(\gamma)$ will meet $M_{r-k}(\varepsilon)$ in an $M_l(\delta)$ and $M_{r-k-l}(\varepsilon')$ in I'_{r-k-l} points, where $I'_{r-k-l}=q\delta_0$. From a proper section,

$$\delta_0=\sigma_{r-k-l}[\alpha\varepsilon]_0+\sigma_{r-k-l-1}[\alpha\varepsilon]_1+\ldots+\sigma_1[\alpha\varepsilon]_{r-k-l-1}+[\alpha\varepsilon]_{r-k-l}.$$

Multiplying by q and using the assumed relations, we have

$$q\delta_0=I'_{r-k-l}=\sigma_{r-k-l}[\alpha\varepsilon']_0+\sigma_{r-k-l-1}[\alpha\varepsilon']_1+\ldots.+\sigma_1[\alpha\varepsilon']_{r-k-l-1}+q[\alpha\varepsilon]_{r-k-l}.$$

Hence, $[\alpha\varepsilon']_{r-k-l}=q[\alpha\varepsilon]_{r-k-l}$.

The theorem (58) for $k=0$ becomes, by the use of (56), the theorem (57). Either will serve as a basis for the following definitions of the residual index numbers of $M_r(\alpha)$.

2. If $n-r-1$ spreads be passed through $M_r(\alpha)$ in S_n, they meet in an $M_{r+1}(\beta)$, which has in common with $M_r(\alpha)$ the manifold $M_r(\alpha)$ itself. The relative incidence numbers of $M_r(\alpha)$ with regard to itself, $[\alpha\alpha]_i$, according to (56), are the ordinary index numbers of $M_r(\alpha)$, α_i, which in this connection we will denote by $\alpha_{0,i}$. Thus $[\alpha\alpha]_i=\alpha_{0,i}$, $(i=0, 1, \ldots, r)$.

If $n-r$ spreads $\sigma(\lambda)$ on $M_r(\alpha)$ meet in a residual $M_r(\beta)$ which cuts $M_r(\alpha)$ in $M_{r-1}(\gamma)$, let $[\alpha\gamma]_i$, $(i=0, \ldots, r-1)$, be the relative incidence numbers of $M_{r-1}(\gamma)$ on $M_r(\alpha)$. If one of the spreads, say that of order λ_1, be multiplied by a spread F of order q, then $M_{r-1}(\gamma)$ is increased by $M'_{r-1}(\gamma')$, the meet of F and $M_r(\alpha)$, and $[\alpha\gamma]_i$ is increased by $[\alpha\gamma']_i$, which, according to (57), is $q\alpha_i=q\alpha_{0,i}$. But $\sigma_1\alpha_{0,i}$ is increased by $q\alpha_{0,i}$, whence $\alpha_{1,i}$, defined by

$$(59)\qquad [\alpha\gamma]_i=\alpha_{1,i}+\sigma_1\alpha_{0,i}+\alpha_{0,i+1}, \qquad (i=0, 1, \ldots, r-1),$$

is independent of the orders λ and can be regarded as as index number attached to the manifold $M_r(\alpha)$. The change in these index numbers due to a change from S_n to S_{n+1} can be obtained from the corresponding change in $[\alpha\gamma]_i$ [see (54)], in σ, and in $\alpha_{0,i}$.

(60) *The residual index numbers of the second rank defined by* $[\alpha\gamma]_i=\alpha_{1,i}+\sigma_1\alpha_{0,i}+\alpha_{0,i+1}$, $(i=0, \ldots, r-1)$, *in terms of* $M_{r-1}(\gamma)$ *are independent of the orders of the spreads which determine* $M_{r-1}(\gamma)$. *The change in them due to a change from* S_n *to* S_{n+1} *is expressed by*

$$\alpha'_{0,i}+\alpha'_{0,i-1}=\alpha_{0,i}, \quad \alpha'_{1,i}+\alpha'_{1,i-1}=\alpha_{1,i}-\alpha_{0,i-1}.$$

In particular, according to (42), $\alpha_{1,0}=0$. For example, let $M_2(\alpha)$ in S_4 be the regular intersection of $u^l=0$, $u^m=0$. Then $\alpha_{0,0}=lm$, $\alpha_{0,1}=-lm(l+m)$, $\alpha_{0,2}=lm(l^2m+lm+m^2)$. Let $u^l f^{\lambda-l}+u^m f^{\lambda-m}=0$, $u^l f^{\mu-l}+u^m f^{\mu-m}=0$ be spreads of orders λ, μ on $M_2(\alpha)$ which meet in $M_2(\beta)$, which cuts $M_2(\alpha)$ in $M_1(\gamma)$. Then $M_1(\gamma)$ is the regular intersection of $u^l=0, u^m=0$, and $\begin{vmatrix} f^{\lambda-l} & f^{\lambda-m} \\ f^{\mu-l} & f^{\mu-m}\end{vmatrix}=0$. According to (57), $[\alpha\gamma]_0=lm(\lambda+\mu-l-m)$ and $[\alpha\gamma]_1=-lm(l+m)(\lambda+\mu-l-m)$. Thus, $\alpha_{1,0}=[\alpha\gamma]_0-(\lambda+\mu)\alpha_{0,0}-\alpha_{0,1}=0$ and $\alpha_{1,1}=[\alpha\gamma]_1-(\lambda+\mu)\alpha_{0,1}-\alpha_{0,2}=l^2m^2$; *i. e.*, $\alpha_{1,0}$ and $\alpha_{1,1}$ are independent of λ, μ.

The residual index numbers of the third rank of $M_r(\alpha)$ are defined as follows: Let $n-r+1$ spreads $\sigma(\lambda)$ on $M_r(\alpha)$ meet in the residual $M_{r-1}(\beta^{(1)})$

which cuts $M_r(\alpha)$ in $M_{r-2}(\gamma^{(1)})$ with residual incidence numbers $[\alpha\gamma^{(1)}]_i$. Then, if $\alpha_{2,i}$ be defined by

$$(61) \qquad [\alpha\gamma^{(1)}]_i=\alpha_{2,i}+\sigma_1\alpha_{1,i}+\alpha_{1,i+1}+\sigma_2\alpha_{0,i}+\sigma_1\alpha_{0,i+1}+\alpha_{0,i+2}, \qquad (i=0, 1, \ldots, r-2),$$

it is independent of the orders and is an index number of $M_r(\alpha)$. For if λ_1 be increased by 1 by adding a linear factor, $M_{r-2}(\gamma^{(1)})$ is increased by a section of the $M_{r-1}(\gamma')$ obtained from $\sigma'(\lambda_2, \ldots, \lambda_{n-r+1})$, $[\alpha\gamma^{(1)}]_i$ is increased by $[\alpha\gamma']_i$, which by the definition of the numbers of the second rank is $\alpha_{1,i}+\sigma'_1\alpha_{0,i}+\alpha_{0,i+1}$. This change in $[\alpha\gamma^{(1)}]_i$ is balanced by the change in the σ's on the right of (61), whence $\alpha_{2,i}$ is unaltered.

The residual numbers of the $(k+1)$-th rank are defined in terms of those of earlier ranks as follows: If $n-r+k-1$ spreads $\sigma(\lambda)$ on $M_r(\alpha)$ meet in a residual $M_{r-k+1}(\beta^{(k-1)})$ which cuts $M_r(\alpha)$ in $M_{r-k}(\gamma^{(k-1)})$, whose residual incidence numbers are $[\alpha\gamma^{(k-1)}]$, then $\alpha_{k,i}$ is defined by

$$\begin{aligned}(62) \qquad [\alpha\gamma^{(k-1)}]_i &=\alpha_{k,i}+(\sigma_1\alpha_{k-1,i}+\alpha_{k-1,i+1})+(\sigma_2\alpha_{k-2,i}+\sigma_1\alpha_{k-2,i+1}+\alpha_{k-2,i+2})\\ &\quad+\cdots+(\sigma_k\alpha_{0,i}+\sigma_{k-1}\alpha_{0,i+1}+\cdots+\alpha_{0,i+k})\\ &=(\alpha_{k,i}+\alpha_{k-1,i+1}+\cdots+\alpha_{0,i+k})+\sigma_1(\alpha_{k-1,i}+\alpha_{k-2,i+1}+\cdots\\ &\quad+\alpha_{0,i+k-1})+\cdots+\sigma_{k-1}(\alpha_{1,i}+\alpha_{0,i+1})+\sigma_k(\alpha_{0,i}),\\ &\qquad (k=0, 1, \ldots, r;\ i=0, 1, \ldots, r-k).\end{aligned}$$

It can be shown, as above, that they are independent of the orders λ, and therefore also of the manifold $M_{r-k}(\gamma^{(k-1)})$. A somewhat more convenient form of the definition can be obtained by introducing the abbreviations

$$(63) \qquad \begin{cases} A_{c,d}=\alpha_{c,d}+\sigma_1\alpha_{c,d-1}+\sigma_2\alpha_{c,d-2}+\cdots+\sigma_d\alpha_{c,0}, \text{ and}\\ \bar{A}_{c,d}=\alpha_{c,d}+\sigma_1\alpha_{c-1,d}+\sigma_2\alpha_{c-2,d}+\cdots+\sigma_c\alpha_{0,d}.\end{cases}$$

(64) *The residual index numbers of $M_r(\alpha)$ of the $(k+1)$-th rank, $\alpha_{k,i}$, $(k=0, \ldots, r;\ i=0, \ldots, r-k)$, are defined by*

$$[\alpha\gamma^{(k-1)}]_i=\sum_{l=0}^{i+k}A_{l,i+k-l}-\sum_{l=k+1}^{i+k}\bar{A}_{l,i+k-l},$$

in terms of the index numbers of lower ranks. They are independent of the orders λ and the manifold $M_{r-k}(\gamma^{(k-1)})$.

To identify the two definitions, note that the coefficient of σ_j on the right in (64) is $\sum_{l=0}^{i+k}\alpha_{l,i+k-l-j}-\sum_{l=1+k}^{i+k}\alpha_{l-j,i+k-l}$, while in the original definition it is $\alpha_{k-j,i}+\cdots+\alpha_{0,i+k-j}$. The difference between the two is

$$\sum_{l=i+k-j+1}^{i+k}\alpha_{l,i+k-l-j}-\sum_{l=k+1}^{i+k}\alpha_{l-j,i+k-l}=\sum_{l=i+k-j+1}^{i+k}\alpha_{l,i+k-j-l}.$$

This last sum vanishes, since in each term the second subscript of the α is neg-

ative. Note that in the definition (64) index numbers of rank greater than k formally appear, but they cancel in the expanded formula.

3. (65) *The complete set of residual index numbers of a regular $M_r(\alpha)$ in S_n, determined by spreads of orders $\lambda_1, \ldots, \lambda_{n-r}$, is given by*

$$\alpha_{k,i} = (-1)^{k+i}\pi_{n-r}[\Sigma_k \Sigma_r - \Sigma_{k-1}\Sigma_{i+1}],$$

where π_{n-r} is the product, and Σ_j the complete symmetric polynomial of degree j, formed from the given orders.

Let $n-r+k-1$ spreads $\sigma(l)$ on $M_r(\alpha)$ meet in a residual $M_{r-k+1}(\beta)$ which cuts $M_r(\alpha)$ in $M_{r-k}(\gamma)$. These spreads can be taken in the form

$$\begin{array}{l} u^{\lambda_1} v^{l_1-\lambda_1} + \ldots \quad + u^{\lambda_{n-r}} v^{l_1-\lambda_{n-r}} \quad = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ u^{\lambda_1} v^{l_{n-r+k-1}-\lambda_1} + \ldots + u^{\lambda_{r-s}} v^{l_{n-r+k-1}-\lambda_{n-r}} = 0, \end{array}$$

where $u^{\lambda_1}=0, \ldots, u^{\lambda_{n-r}}=0$ cut out $M_r(\alpha)$. Then $M_{r-k}(\gamma)$ is the regular intersection of $M_r(\alpha)$ by the spread defined by the matrix

$$\left\| \begin{array}{lll} v^{l_1-\lambda_1}, & \ldots, & v^{l_1-\lambda_{n-r}} \\ \ldots & \ldots & \ldots \\ v^{l_{n-r+k-1}-\lambda_1}, & \ldots, & v^{l_{n-r+k-1}-\lambda_{n-r}} \end{array} \right\| = 0,$$

whose order, according to Salmon's formula,* is

$$(\sigma_k - \sigma_{k-1}\Sigma_1 + \sigma_{k-2}\Sigma_2 - \ldots + (-1)^{k-1}\sigma_1\Sigma_{k-1} + (-1)^k\Sigma_k).$$

From (57) and (64) we have, respectively,

$$[\alpha\gamma^{(k-1)}]_i = (\sigma_k - \sigma_{k-1}\Sigma_1 + \ldots + (-1)^k\Sigma_k)\alpha_i, \text{ and}$$

$$[\alpha\gamma^{(k-1)}]_i = \sum_{l=0}^{i+k} A_{l,\,i+k-l} - \sum_{l=k+1}^{k+i} \bar{A}_{l,\,i+k-l}.$$

Equating the terms in σ_{k-j}, we find that

$$(-1)^j\Sigma_j\alpha_i = \sum_{l=0}^{i+k}\alpha_{l,\,i-l+j} - \sum_{l=k+1}^{k+i}\alpha_{l-k+j,\,i+k-l}, \quad (j=0, 1, \ldots, k).$$

Writing this equality for $j=k$, $i=i$, and again for $j=k-1$, $i=i+1$, we get, by subtraction, $(-1)^k(\Sigma_k\alpha_i + \Sigma_{k-1}\alpha_{i+1}) = \alpha_{k,i}$. According to [(11), "R. S.," I],

$$\alpha_i = (-1)^i\pi_{n-r}\Sigma_i, \quad \alpha_{i+1} = (-1)^{i+1}\pi_{n-r}\Sigma_{i+1},$$

whence $\alpha_{k,i} = (-1)^{i+k}\pi_{n-r}(\Sigma_k\Sigma_i - \Sigma_{k-1}\Sigma_{i+1})$.

4. In the above particular case of a regular $M_r(\alpha)$ the relations $\alpha_{k,i} + \alpha_{i+1,k-1} = 0$, $\alpha_{k+1,k} = 0$ exist. Let us prove generally that

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* A simple proof of this formula is given in § 5.

(66) *The $\frac{1}{2}r(r+1)$ residual index numbers of $M_r(\alpha)$ satisfy the following relations: $\alpha_{k,i}+\alpha_{i+1,k-1}=0$, and $\alpha_{k,k-1}=0$.*

Let $n-r+k-1$ spreads $\sigma(\lambda)$ on $M_r(\alpha)$ determine $M_{r-k}(\gamma^{(k-1)})$, and $n-k-1$ spreads $\tau(\mu)$ on $M_r(\alpha)$ determine $M_k(\gamma^{(r-k-1)})$. If these two manifolds on $M_r(\alpha)$ meet in I_{r-k} points, then we have from (53), if we use first $M_{r-k}(\gamma^{(k-1)})$ as $M(\varepsilon)$ and second $M_k(\gamma^{(r-k-1)})$ as $M(\varepsilon)$, the equations:

$$\begin{aligned} I_{r-k} &= [\alpha\gamma^{(k-1)}]_{r-k}+\tau_1[\alpha\gamma^{(k-1)}]_{r-k-1}+\ldots+\tau_{r+k}[\alpha\gamma^{(k-1)}]_0, \\ &= [\alpha\gamma^{(r-k-1)}]_k+\sigma_1[\alpha\gamma^{(r-k-1)}]_{k-1}+\ldots+\sigma_k[\alpha\gamma^{(r-k-1)}]_0. \end{aligned}$$

Here the $[\alpha\gamma^{(k-1)}]_j$ are given by (64) in terms of the σ's, and the $[\alpha\gamma^{(r-k-1)}]_j$ are given in terms of the τ's. Equating coefficients of $\sigma_s\tau_t$, we get

$$\sum_{l=0}^{r-t}\alpha_{l,\,r-l-s-t}-\sum_{l=k+1}^{r-t}\alpha_{l-s,\,r-l-t}=\sum_{l=0}^{r-s}\alpha_{l,\,r-l-s-t}-\sum_{l=r-k+1}^{r-s}\alpha_{l-t,\,r-l-s},$$

where $0\leqq t\leqq r-k$ and $0\leqq s\leqq k$. If in this equality t be increased by 1 and s be diminished by 1, and the result be subtracted from the given equality, we get $\alpha_{r-t,-s}-\alpha_{k+1-s,\,r-t-k-1}=-\alpha_{r-s+1,\,-t-1}+\alpha_{r-k-t,\,k-s}$, where now $0\leqq t\leqq r-k-1$ and $1\leqq s\leqq k$. Thus two of the α's have a negative subscript and vanish, and $\alpha_{j,m}+\alpha_{m+1,j-1}=0$, $(j=k,\ldots,1;\ m=r-k-1,\ldots,0)$.

5. There is one case in which the residual index numbers all can be expressed in terms of the $r+1$ ordinary index numbers.

(67) *The residual index numbers of an $M_{n-2}(\alpha)$ in S_n are given in terms of the ordinary index numbers by the formula $\alpha_{h,l}=\alpha_{h-1}\alpha_{l-1}-\alpha_{h-2}\alpha_l$, where $h=1,2,\ldots,n-2$ and $l=0,1,\ldots,n-h-2$, while $\alpha_{-1}=0$.*

Using the above notation for $n-r=2$, the spreads $\sigma(\lambda)$ meet in a residual $M_{n+k-1}(\beta)$ and the spreads $\tau(\mu)$ in a residual $M_{k+1}(\beta')$. The two residual spreads meet in $\beta_0\beta_0'$ points, where

$$\begin{aligned} &\beta_0=\sigma_{k+1}-A_{k-1}, \text{ and} \\ &\beta_0'=\tau_{n-k-1}-\alpha_0\tau_{n-k-3}-\alpha_1\tau_{n-k-4}-\ldots-\alpha_{n-k-4}\tau_1-\alpha_{n-k-3}. \end{aligned}$$

These common points consist of the O points common to the spreads $\sigma(\lambda)$ and $\tau(\mu)$ outside of $M_{n-2}(\alpha)$, and of the I points common to the $M_{n-k-2}(\gamma^{(k-1)})$ and the $M_k(\gamma^{(n-k-3)})$ on $M_{n-2}(\alpha)$. Here

$$\begin{aligned} O&=\sigma_{k+1}\tau_{n-k-1}-\tau_{n-k-1}A_{k-1}-\tau_{n-k-2}A_k-\ldots-\tau_1A_{n-3}-A_{n-2}, \text{ and} \\ I&=[\alpha\gamma^{(k-1)}]_0\tau_{n-k-2}+[\alpha\gamma^{(k-1)}]_1\tau_{n-k-3}+\ldots+[\alpha\gamma^{(k-1)}]_{n-k-3}\tau_1+[\alpha\gamma^{(k-1)}]_{n-k-2}. \end{aligned}$$

By equating the coefficients of τ in $\beta_0\beta_0'=O+I$, we get

$$\alpha_{l-1}(A_{k-1}-\sigma_{k+1})=-A_{k+l}+[\alpha\gamma^{(k-1)}]_l, \quad (l=0,1,\ldots,n-k-2).$$

In this, after dropping the obvious equalities found from the coefficients of σ_k and σ_{k+1}, we find from the coefficient of σ_j that

$$\alpha_{l-1}\cdot\alpha_{k-j-1}=-\alpha_{k+l-j}+\alpha_{k-j,\,l}+\alpha_{k-j-1,\,l+1}+\ldots\ldots+\alpha_{0,\,k-j+l},$$

where $j=0,1,\ldots\ldots,k-1$. Putting $k-j=h$, we have

$$\alpha_{h-1}\ \ \alpha_{l-1}=-\alpha_{h+l}+\alpha_{h,\,l}+\alpha_{h-1,\,l+1}+\ldots\ldots+\alpha_{0,\,h+l},$$
$$(l=0,\ldots\ldots,n-k-2;\ h=1,2,\ldots\ldots,k).$$

Allowing h to diminish by 1 and l to increase by 1 and subtracting, we get $\alpha_{h,\,l}=\alpha_{h-1}\,\alpha_{l-1}-\alpha_{h-2}\,\alpha_l$, which is also true when $h=1$ if $\alpha_{-1}=0$.

As a verification let $M_{n-2}(\alpha)$ be regular, being cut out by spreads of orders λ_1,λ_2. Then $\alpha_{h,\,l}=(-1)^{h+l}\lambda_1^2\lambda_2^2(\Sigma_{h-1}\Sigma_{l-1}-\Sigma_{h-2}\Sigma_l)$. But for this case we had found in (65) that $\alpha_{h,\,l}=(-1)^{h+l}\lambda_1\lambda_2(\Sigma_h\Sigma_l-\Sigma_{h-1}\Sigma_{l+1})$. To identify these, we notice by direct multiplication that $\Sigma_h\Sigma_l-\Sigma_{h-1}\Sigma_{l+1}=(\lambda_1\lambda_2)^{h-1}\Sigma_{l-h}$ if $h\overline{<}l$. Hence, $\Sigma_{h-1}\Sigma_{l-1}-\Sigma_{h-2}\Sigma_l=(\lambda_1\lambda_2)^{h-2}\Sigma_{l-h}$, and the two expressions are reconciled.

6. The generalized problem in restricted systems has, in the case of a common curve, the following solution:

(68) *If in S_n an $M_{r_1}(\alpha^{(1)}),\ldots\ldots,M_{r_i}(\alpha^{(i)})$, where $r_1+\ldots\ldots+r_i=(i-1)n$, have in common an $M_1(\gamma)$, they meet outside $M_1(\gamma)$ in*

$$O_1=\alpha_0^{(1)}\cdot\alpha_0^{(2)}\cdot\ldots\ldots\cdot\alpha_0^{(i)}-\gamma_0\,\sigma_1\Big(\frac{-[\alpha^{(k)}\gamma]_1}{\gamma_0}\Big)-\gamma_1\ \textit{points}.$$

Let spreads $\sigma^{(k)}(\lambda_1^{(k)},\ldots\ldots,\lambda_{n-r_k}^{(k)})$ on $M_{r_k}(\alpha^{(k)})$ meet again in $M_{r_k}(\beta^{(k)})$, which meets $M_1(\gamma)$ in $I_1^{(k)}=\sigma_1^{(k)}\gamma_0+[\alpha^{(k)}\gamma]_1$ points, where $\beta_0^{(k)}=\pi(\lambda^{(k)})-\alpha_0^{(k)}=\pi_k-\alpha_0^{(k)}$. All n of the spreads on $M_1(\gamma)$ meet outside $M_1(\gamma)$ in

$$\Omega_1=\pi_1\pi_2\ldots\ldots\pi_i-\gamma_0(\sigma_1^{(1)}+\sigma_2^{(1)}+\ldots\ldots+\sigma_i^{(1)})-\gamma_1\ \text{points}.$$

The Ω_1 points are made up of the O_1 points common to $M_{r_1}(\alpha^{(1)}),\ldots\ldots,M_{r_i}(\alpha^{(i)})$; of the $(\pi_1-\alpha_0^{(1)})\alpha_0^{(2)}\ldots\ldots\alpha_0^{(i)}-I_1^{(1)}$ points common to $M_{r_1}(\beta^{(1)})$, $M_{r_2}(\alpha^{(2)}),\ldots\ldots,$ $M_{r_i}(\alpha^{(i)})$, etc.; of the $(\pi_1-\alpha_0^{(1)})(\pi_1-\alpha_0^{(2)})\alpha_0^{(3)}\ldots\ldots\alpha_0^{(i)}$ points common to $M_{r_1}(\beta^{(1)})$, $M_{r_2}(\beta^{(2)})$, $M_{r_3}(\alpha^{(3)}),\ldots\ldots,M_{r_i}(\alpha^{(i)})$, etc.; ....; finally of the $(\pi_1-\alpha_0^{(1)})(\pi_2-\alpha_0^{(2)})$ $\ldots\ldots(\pi_i-\alpha_0^{(i)})$ points common to $M_{r_1}(\beta^{(1)}),\ldots\ldots,M_{r_i}(\beta^{(i)})$. Thus

$$\Omega_1=O_1+\Sigma\{(\pi_1-\alpha_0^{(1)})\alpha_0^{(2)}\ldots\ldots\alpha_0^{(i)}-I_1\}$$
$$+\Sigma\{(\pi_1-\alpha_0^{(1)})(\pi_1-\alpha_0^{(2)})\alpha_0^{(3)}\ldots\ldots\alpha_0^{(i)}\}+\ldots\ldots+\pi(\pi_1-\alpha_0^{(1)}).$$

By using the identity

$$x_1x_2\ldots\ldots x_i=[(x_1-y_1)+y_1][(x_2-y_2)+y_2]\ldots\ldots[(x_i-y_i)+y_i]$$
$$=y_1y_2\ldots\ldots y_i+\Sigma(x_1-y_1)y_2\ldots\ldots y_i$$
$$+\Sigma(x_1-y_2)(x_2-y_2)y_3\ldots\ldots y_i+\ldots\ldots+\pi(x_1-y_1),$$

we find that

$$\Omega_1=O_1-\Sigma I_1^{(1)}+\pi_1\pi_2\ldots\ldots\pi_i-\alpha_0^{(1)}\alpha_0^{(2)}\ldots\ldots\alpha_0^{(i)}$$
$$=\pi_1\pi_2\ldots\ldots\pi_i-\gamma_0[\sigma_1^{(1)}+\ldots\ldots+\sigma_1^{(i)}]-\{[\alpha^{(1)}\gamma]_1+\ldots\ldots$$
$$+[\alpha^{(i)}\gamma]_1\}-\alpha_0^{(1)}\alpha_0^{(2)}\ldots\ldots\alpha_0^{(i)}+O_1.$$

Therefore $O_1=\alpha_0^{(1)}\alpha_0^{(2)}\ldots\alpha_0^{(i)}-\gamma_0\left\{-\frac{[\alpha^{(1)}\gamma]_1}{\gamma_0}-\ldots-\frac{[\alpha^{(i)}\gamma]_1}{\gamma_0}\right\}-\gamma_1.$

This formula for O_1 is like the usual formula, except that in forming σ_1 the given orders are replaced by $-\frac{[\alpha^{(k)}\gamma]_1}{\gamma_0}=-\frac{[\alpha^{(k)}\gamma]_1}{[\alpha^{(k)}\gamma]_0}$.

The relative incidence numbers $[\alpha^k\gamma]_1$ can be replaced by relative index numbers according to the following theorem, but the formula for O_1 then loses its resemblance to the original formula.

(69) *For an $M_1(\gamma)$ on $M_r(\alpha)$, $\gamma_0=(\alpha\gamma)_0=[\alpha\gamma]_0$ and $\gamma_1=(\alpha\gamma)_1+[\alpha\gamma]_1$.*

Only the last equality needs proof, the others being a matter of definition. Let $n-r$ spreads $\sigma(\lambda)$ on $M_r(\alpha)$ meet in a residual $M_r(\beta)$ which cuts $M_r(\alpha)$ in $M_{r-1}(\varepsilon)$, which in turn meets $M_1(\gamma)$ in $I_1=\sigma_1\gamma_0+[\alpha\gamma]_1$ points. Then r further spreads $\tau(\mu)$ on $M_1(\gamma)$ determine an $M_{n-r}(\zeta)$ which meets $M_r(\alpha)$ and $M_r(\beta)$ in O_1 points outside of $M_1(\gamma)$. Here

$$O_1=\sigma_{n-r}\tau_r-\gamma_0(\sigma_1+\tau_1)-\gamma_1=\{\tau_r\alpha_0-\gamma_0\tau_1-(\alpha\gamma)_1\}+\{\tau_r\beta_0-I_1\}.$$

Since $\alpha_0+\beta_0=\sigma_{n-r}$, this leads to $\gamma_1=(\alpha\gamma)_1+[\alpha\gamma]_1$.

7. From (68) and (69) the undetermined index number β_2 or $(\alpha\gamma)_1$ of (51) can be obtained when $r=n-2$. For then $M_{n-2}(\alpha)$ and $M_2(\beta)$ meet in $M_1(\gamma)$ and no further points, whence, from (68),

$$O_1=0=\alpha_0\beta_0-\gamma_0\left(-\frac{[\alpha\gamma]_1}{\gamma_0}-\frac{[\beta\gamma]_1}{\gamma_0}\right)-\gamma_1.$$

Replacing $[\alpha\gamma]_1$ and $[\beta\gamma]_1$ according to (69), we have

$$0=\alpha_0\beta_0+\gamma_1-(\alpha\gamma)_1-(\beta\gamma)_1.$$

(70) *The undetermined index number β_2 or $(\alpha\gamma)_1$ of* (51) *is found in the case of an $M_{n-2}(\alpha)$ from either*

$$\alpha_0\beta_0=(\alpha\gamma)_1+(\beta\gamma)_1-\gamma_1 \text{ or } B_2+\left\{\binom{n-3}{2}+1\right\}A_{n-2}=\alpha_0\beta_0+\gamma_1.$$

§ 5. *Manifolds Defined by Matrices.*

1. In order that the various terms in the expansion of a determinant or a subdeterminant of a matrix, $M_{n+k,n}$, with n rows and $n+k$ columns, whose elements are forms in $d+1$ variables, may be homogeneous, it is necessary that the orders of the elements be taken as indicated in the array

$$M_{n+k,n}=\left\|\begin{matrix} l_1+\lambda_1 & l_2+\lambda_1 & \ldots & l_{n+k}+\lambda_1 \\ l_1+\lambda_2 & l_2+\lambda_2 & \ldots & l_{n+k}+\lambda_2 \\ \ldots & \ldots & \ldots & \ldots \\ l_1+\lambda_n & l_2+\lambda_n & \ldots & l_{n+k}+\lambda_n \end{matrix}\right\|.$$

If the $d+1$ variables are linear in the $c+1$ coordinates of an S_c, the vanishing of the matrix defines a manifold of dimension $c-(k+1)$ in S_c. For if the matrix of the first $n-1$ columns vanishes on a certain manifold M', the $k+1$ spreads obtained by adding each of the remaining columns to form a determinant all contain M' and meet in a residual $M_{c-(k+1)}$ which cuts M' in an $M_{c-(k+2)}$. Since for the general point of $M_{c-(k+1)}$ M' does not vanish, the vanishing of the $k+1$ determinants entails the vanishing of the matrix. In this section the first three index numbers of $M_{c-(k+1)}$ in S_c or of $M_{n+k,\,n}$ are derived, all the index numbers of $M_{n+1,\,n}$ are obtained, and a tentative formula for the general index number of $M_{n+k,\,n}$ which holds for the first $k+3$ numbers is given. A formula for the number of linear spaces which meet a prescribed number of given linear spaces is obtained in terms of the index numbers of $M_{n+k,\,n}$. Owing to the limitation of the tentative formula, this number is evaluated only for the case when the given spaces are lines and the case when the given spaces are nine planes in S_5.

2. Throughout this section we denote respectively by μ_i and $\bar{\mu}_i$ the elementary and the complete symmetric functions of degree i formed from $l_1, \ldots, l_{n+k}$; and by ν_i and $\bar{\nu}_i$, respectively, the complete and the elementary symmetric functions of $\lambda_1, \ldots, \lambda_n$. Further, let

$$(71) \qquad \begin{cases} H_j = \mu_j + \mu_{j-1}\nu_1 + \mu_{j-2}\nu_2 + \ldots + \nu_j, \text{ and} \\ \overline{H}_j = \bar{\mu}_j + \bar{\mu}_{j-1}\bar{\nu}_1 + \bar{\mu}_{j-2}\bar{\nu}_2 + \ldots + \bar{\nu}_j; \end{cases}$$

and let $m_{n+k,\,n\,;\,j}$, $(j=0, 1, \ldots)$, be the index numbers of $M_{n+k,\,n}$. From the well-known relations

$$\begin{aligned} m_j &= \mu_j - \mu_{j-1}\bar{\mu}_1 + \mu_{j-2}\bar{\mu}_2 - \ldots + (-1)^j \bar{\mu}_j = 0, \text{ and} \\ n_j &= \nu_j - \nu_{j-1}\bar{\nu}_1 + \nu_{j-2}\bar{\nu}_2 - \ldots + (-1)^j \bar{\nu}_j = 0, \text{ there follows} \\ h_j &= H_j - H_{j-1}\overline{H}_1 + H_{j-2}\overline{H}_2 - \ldots + (-1)^j \overline{H}_j = 0. \end{aligned}$$

For if we compare h_j with $m_j + m_{j-1}n_1 + m_{j-2}n_2 + \ldots + n_j$, which is evidently zero, we find that $h_j = \sum_{r=0}^{j} (-1)^r H_{j-r}\overline{H}_r = \sum_{r,\,s,\,t} (-1)^r \mu_{j-r-s}\bar{\mu}_{r-t}\nu_s\bar{\nu}_t$, while

$$\sum_{r=0}^{j} m_{j-r}n_r = \sum_{r=0}^{j}\Big(\sum_{s=0}^{j-r}(-1)^s \mu_{j-r-s}\bar{\mu}_s\Big)\Big(\sum_{t=0}^{r}(-1)^t \nu_{r-t}\bar{\nu}_t\Big) = \sum_{r,\,s,\,t}(-1)^{s+t}\mu_{j-r-s}\bar{\mu}_s\nu_{r-t}\bar{\nu}_t.$$

If we set $r-t=s'$, $s=r'-t'$, $t=t'$, the first sum reduces to the second.

The following equations are fairly obvious:

$$(72) \qquad H_j^{0,\,0} = H_j^{1,\,0} + l_1 H_{j-1}^{1,\,0}, \quad H_j^{0,\,0} = \lambda_1 H_{j-1}^{0,\,0} + H_j^{0,\,1}, \quad H_j^{1,\,0} = \lambda_1 H_{j-1}^{1,\,0} + H_j^{1,\,1},$$

where the first superscript 1 or 0 refers to the omission or retention of l_1 in the formation of H, and the second superscript indicates similarly the omission or retention of λ_1.

3. With the aid of these formulæ an immediate proof* of Salmon's formula for the order of a matrix can be given.

(73) *The order of a matrix* $M_{n+k, n}$ *is* H_{k+1}.

The theorem is obviously true for the matrix $M_{k, 1}$; let us assume it to be true for all matrices $M_{r, s}$ such that $r+s<2n+k$. Let $M_{n+k-1, n}$ be the matrix of order $m_{n+k-1, n}$ obtained by dropping the first column of $M_{n+k, n}$; and let $M_{n+k-1, n-1}$ of order $m_{n+k-1, n-1}$ be that obtained by dropping the first row and column. Consider the section of the manifold $M_{n+k, n}=0$ by the spreads of orders $l_1+\lambda_2, \ldots, l_1+\lambda_n$ in the first column, $\pi=(l_1+\lambda_2)\cdot(l_1+\lambda_3)\cdot\ldots\cdot(l_1+\lambda_n)$. This section is regular and of order $\pi m_{n+k, n}$. It breaks up into two partial sections. For the one partial section the spread of order $l_1+\lambda_1$ and $M_{n+k-1, n}$ vanish, whence its order is $\pi\cdot(l_1+\lambda_1)\cdot m_{n+k-1, n}$. For the other partial section the spread of order $l_1+\lambda_1$ does not vanish, while $M_{n+k-1, n-1}$ does vanish, whence its order is $\pi\cdot m_{n+k-1, n-1}$. Therefore $m_{n+k, n}=(l_1+\lambda_1)m_{n+k-1, n}+m_{n+k-1, n-1}$. Both orders on the right are known from the assumed formula, whence $m_{n+k, n}=(l_1+\lambda_1)H_k^{1, 0}+H_{k+1}^{1, 1}$. By using (72) we find that $m_{n+k, n}=l_1 H_k^{1, 0}+H_{k+1}^{1, 0}=H_{k+1}^{0, 0}=H_{k+1}$, which proves the theorem.

4. Similar considerations readily lead to the following theorem:

(74) *The curve (in* S_{k+2}*) of order* H_{k+1} *defined by the matrix* $M_{n+k, n}=0$ *has the second index number* $m_{n+k, n; 1}=-kH_{k+2}-H_1 H_{k+1}$ *and the genus* p *determined by* $2p-2=kH_{k+2}+H_1 H_{k+1}-(k+3)H_{k+1}$.

The value of the genus is obtained at once from that of $m_{n+k, n; 1}$ by using [(27), "R. S.," I]. The first and second index numbers of the matrix

$$M_{1+k, 1}=\|l_1+\lambda_1, \ldots, l_{1+k}+\lambda_1\|=0,$$

a regular spread, are

$$m_{1+k, 1; 0}=(l_1+\lambda_1)\cdot\ldots\cdot(l_{1+k}+\lambda_1)$$

and

$$m_{1+k, 1; 1}=-(l_1+\ldots+l_{1+k}+(k+1)\lambda_1)m_{1+k, 1; 0}=-(k\lambda_1+H_1)H_{k+1}.$$

Since $H_{k+2}^{0, 0}=\lambda_1 H_{k+1}^{0, 0}+H_{k+2}^{0, 1}$ and $H_{k+2}^{0, 1}=\mu_{k+2}=0$, the value of $m_{1+k, 1; 1}$ given in (74) also reduces to $-(k\lambda_1+H_1)H_{k+1}$. We assume, then, that (74) is true for matrices $M_{r, s}$ such that $r+s<2n+k$, and proceed as above. If Σ_1 denote $(l_1+\lambda_2)+\ldots+(l_1+\lambda_n)$, the second index number of the section of $M_{n+k, n}$, according to [(18), "R. S.," I], is $\pi\{m_{n+k, n; 1}-\Sigma_1 m_{n+k, n; 0}\}$; of the one partial section is $\pi(l_1+\lambda_1)\{m_{n+k-1, n; 1}-(\Sigma_1+l_1+\lambda_1)m_{n+k-1, n; 0}\}$; and of the other par-

* This formula, inferred by Salmon ("Higher Algebra") from a number of special cases, was proved for the first time by Prof. F. F. Decker. This proof will appear in a subsequent number of this JOURNAL.

tial section is $\pi\{m_{n+k-1,\,n-1;\,1}-\Sigma_1 m_{n+k-1,\,n;\,0}\}$. Furthermore, the two partial sections meet in $(l_1+\lambda_1)\pi m_{n+k-1,\,n-1;\,0}$ points. Using the formula (42) for the second index number of the composite curve, we have, after factoring out π,

$$m_{n+k,\,n;\,1}-\Sigma_1 m_{n+k,\,n;\,0}=(l_1+\lambda_1)\{m_{n+k-1,\,n;\,1}-(\Sigma_1+l_1+\lambda_1)m_{n+k-1,\,n;\,0}\}$$
$$+\{m_{n+k-1,\,n-1;\,0}-\Sigma_1 m_{n+k-1,\,n-1;\,0}\}-2(l_1+\lambda_1)m_{n+k-1,\,n-1;\,0}.$$

The terms in Σ_1 cancel, due to the equation connecting the orders. By means of the same equation the term containing $(l_1+\lambda_1)^2$ can be eliminated. Then

$$m_{n+k,\,n;\,1}=(l_1+\lambda_1)\{m_{n+k-1,\,n;\,1}-m_{n+k,\,n;\,0}-m_{n+k-1,\,n-1;\,0}\}+m_{n+k-1,\,n-1;\,1}.$$

Thus, in order to prove (74) we have only to verify that

$$kH^{0,\,0}_{k+2}+H^{0,\,0}_{1}H^{0,\,0}_{k+1}=(l_1+\lambda_1)\{(k-1)H^{1,\,0}_{k+1}+H^{1,\,0}_{1}H^{1,\,0}_{k}+H^{0,\,0}_{k+1}+H^{1,\,1}_{k+1}\}$$
$$+\{kH^{1,\,1}_{k+2}+H^{1,\,1}_{1}H^{1,\,1}_{k+1}\}$$

is true. The terms containing k as a factor vanish, due to the relation

$$H^{0,\,0}_{j}=(l_1+\lambda_1)H^{1,\,0}_{j-1}+H^{1,\,1}_{j}$$

used in the proof of Salmon's formula. The remaining terms are

$$H^{0,\,0}_{1}H^{0,\,0}_{k+1}=(l_1+\lambda_1)[-H^{1,\,0}_{k+1}+H^{0,\,0}_{k+1}+H^{1,\,1}_{k+1}+H^{1,\,0}_{1}H^{1,\,0}_{k}]+H^{1,\,1}_{1}H^{1,\,1}_{k+1}.$$

In the bracket, $-H^{1,\,0}_{k+1}+H^{0,\,0}_{k+1}=l_1H^{1,\,0}_{k}$ and $l_1H^{1,\,0}_{k}+H^{1,\,0}_{1}H^{1,\,0}_{k}=H^{0,\,0}_{1}H^{1,\,0}_{k}$, whence $H^{0,\,0}_{1}H^{0,\,0}_{k+1}=(l_1+\lambda_1)[H^{0,\,0}_{1}H^{1,\,0}_{k}+H^{1,\,1}_{k+1}]+H^{1,\,1}_{1}H^{1,\,1}_{k+1}$. The two terms in $H^{1,\,1}_{k+1}$ reduce to $H^{0,\,0}_{1}H^{1,\,1}_{k+1}$, so that $H^{0,\,0}_{1}$ factors out, leaving $H^{0,\,0}_{k+1}=(l_1+\lambda_1)H^{1,\,0}_{k}+H^{1,\,1}_{k+1}$, which is true and completes the proof of (74).

5. In order to obtain the third index number of $M_{n+k,\,n}$, we first derive all the index numbers $m_{n-1,\,n;\,j}$ of the matrix $M_{n-1,\,n}$ obtained by using the first $n-1$ columns of $M_{n+k,\,n}$. The fact that the number of columns of $M_{n-1,\,n}$ is less than the number of rows requires, according to the conventions in paragraph 2, that the elementary and complete symmetric polynomials be interchanged (so far as rows and columns are concerned); and with this in mind we apply the μ, ν, H notation to $M_{n-1,\,n}$. It is an M_{k-1} in S_{k+1}, and on it we have $k+1$ spreads $\sigma(r)$ of orders $r_0=H_1+l_n,\ldots,r_k=H_1+l_{n+k}$, which meet outside of $M_{n-1,\,n}=0$ in the points of S_{k+1} determined by $M_{n+k,\,n}=0$. This number is given by Salmon's formula (73), which, with our present notation and the use of τ for the elementary symmetric polynomials in $l_n,\ldots,l_{n+k}$, takes the form

$$(\bar\nu_{k+1}+\bar\nu_k\tau_1+\ldots+\tau_{k+1})+\bar\mu_1(\bar\nu_k+\bar\nu_{k-1}\tau_1+\ldots+\tau_k)+\ldots+\bar\mu_k(\bar\nu_1+\tau_1)+\bar\mu_{k+1}$$
$$=\bar{\bar H}_{k+1}+\bar{\bar H}_k\tau_1+\bar{\bar H}_{k-1}\tau_2+\ldots+\tau_{k+1}.$$

On the other hand, this number is given by

$$\sigma_{k+1}-m_{n-1,\,n;\,0}\sigma_{k-1}-m_{n-1,\,n;\,1}\sigma_{k-2}-\ldots-m_{n-1,\,n;\,k-1}.$$

Equating coefficients of τ_{k+1-j}, we get

$$(75)\quad \overline{H}_j = H_1^j - \binom{j}{2} H_1^{j-2}\, m_{n-1,\,n\,;\,0} - \binom{j}{3} H_1^{j-3} m_{n-1,\,n\,;\,0} - \ldots\ldots - \binom{j}{j} m_{n-1,\,n\,;\,j-2}.$$

If we begin with the obvious equations $1=1$ and $\overline{H}_1 = H_1$, and take also the $j-1$ equations (75) for $j=2, 3, \ldots., j$, and if we multiply this system of $j+1$ equations in order, beginning with the last, by

$$\binom{j}{0},\ -\binom{j}{1}H_1,\ \binom{j}{2}H_1^2,\ \ldots.,\ (-1)^j H_1^j,$$

respectively, and add, we find that

$$(76)\quad m_{n-1,\,n\,;\,j-2} = (-1)^{j-1}[H_1^j - \binom{j}{1} H_1^{j-1}\overline{H}_1 + \binom{j}{2} H_1^{j-2}\overline{H}_2 \ldots. (-1)^j \binom{j}{j}\overline{H}_j],$$

due to the binomial formula

$$(77)\quad S = \binom{a}{0}\binom{a+c}{b} - \binom{a}{1}\binom{a+c-1}{b} + \binom{a}{2}\binom{a+c-2}{b} - \ldots. + (-1)^{a+c-b}\binom{a}{a+c-b} = \binom{c}{b-a},$$

with the particular cases: $S=1$ if $b=a$, or if $b=a+c$; $S=0$ if $b=0$, or if $b<a$, or if $b>a+c$, or if $c=0$, unless at the same time $b=a$. Hence,

(78) *The index numbers of*

$$M_{n-1,\,n} = \left\| \begin{matrix} l_1+\lambda_1, & \ldots., & l_{n-1}+\lambda_1 \\ \ldots & \ldots & \ldots \\ l_1+\lambda_n, & \ldots., & l_{n-1}+\lambda_n \end{matrix} \right\|$$

are given by (76), *where* $H_1 = \overline{H}_1$, $\overline{H}_i = \bar{\mu}_i + \bar{\mu}_{i-1}\bar{\nu}_1 + \ldots. + \bar{\nu}_i$, *and* $\bar{\mu}_i (\bar{\nu}_i)$ *are the complete (elementary) symmetric functions of* $\lambda_1, \ldots., \lambda_n$ $(l_1, \ldots., l_{n-1})$.

6. This result will be applied now to the determination of $m_{n+k,\,n\,;\,2}$. In S_{k+3}, $M_{n+k,\,n}=0$ defines an M_2, the residual intersection of the above $\sigma(r)$ spreads on $M_{n-1,\,n}=0$. From (51) and (70) we get the following equations:

$$1°.\quad \begin{cases} m_{n+k,\,n\,;\,0} = \sigma_{k+1} - A_{k-1}, \\ m_{n+k,\,n\,;\,1} = -\sigma_1\, m_{n+k,\,n\,;\,0} + kA_k, \\ m_{n+k,\,n\,;\,2} = (m_{n-1,\,n\,;\,0} - \sigma_2)\, m_{n+k,\,n\,;\,0} - \sigma_1(m_{n+k,\,n\,;\,1} + A_k) - \binom{k+1}{2} A_{k+1}. \end{cases}$$

Here the index numbers in A_j are obtained from (76), in which hereafter the H's will be primed to distinguish them from the H's formed for $M_{n+k,\,n}$. Let us first derive the value of A_j. From the system of equations

$$\sigma_0=\binom{k+1}{0},$$
$$\sigma_1=\binom{k+1}{1}H'_1+\binom{k}{0}\tau_1,$$
$$\dots\dots\dots\dots\dots,$$
$$\sigma_j=\binom{k+1}{j}H_1'^{j}+\binom{k}{j-1}\tau_1 H_1'^{j-1}+\dots+\binom{k-j+2}{1}\tau_{j-1}H'_1+\binom{k-j+1}{0}\tau_j;$$

and the following system obtained from (76) together with two obvious equations adjoined for convenience in summation,

$$m_{n-1,\,n\,;\,j}=(-1)^{j+1}\{H_1'^{j+2}-\binom{j+2}{1}H_1'^{j+1}\overline{H}'_1+\binom{j+2}{2}H_1'^{j}\overline{H}'_2-\dots(-1)^{j}\binom{j+2}{j+2}\overline{H}'_{j+2}\},$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots,$$
$$m_{n-1,\,n\,;\,0}=(-1)^{1}\{H_1'^{2}-\binom{2}{1}H'_1\overline{H}'_1+\binom{2}{2}\overline{H}'_2\},$$
$$m_{n-1,\,n\,;\,-1}=(-1)^{0}\{H'_1-\binom{1}{1}\overline{H}'_1\}=0,$$
$$m_{n-1,\,n\,;\,-2}=(-1)^{-1}\{H_1'^{0}\}+1=0;$$

we find by applying (77) that

$$A_j=\sum_{i=0}^{j+2}\sigma_i\, m_{n-1,\,n\,;\,j-i}=\sigma_{j+2}-\Sigma(-1)^{j-r-s}\binom{j+1-k}{r+s-k-1}H_1'^{\,j+2-r-s}\overline{H}'_s,$$

where in Σ $r=0,\dots,j+2$ and $s=0,\dots,j+2$, while $r+s\overline{<}j+2$. Note that in Σ r, s occur only in the combination $r+s$, except with $\tau_r\overline{H}'_s$, and that $\sum_{r+s=c}\tau_r\overline{H}'_s=H_{r+s}$. If, then, $r+s$ be replaced by t, we get

$$\sigma_{j+2}-A_j=\sum_{t=0}^{j+2}(-1)^{j-t}\binom{j+1-k}{t-k-1}H_tH_1'^{\,k+1-t}.$$

We are interested in the values $j=k-1$, k, $k+1$ only. When $j=k-1$,

$$\sigma_{k+1}-A_{k-1}=\sum_{t=0}^{k+1}(-1)^{k-1-t}\binom{0}{t-k-1}H_tH_1'^{\,k+1-t}.$$

The binomial coefficient is 1 when $t=k+1$, otherwise it is zero, whence

2°. $\sigma_{k+1}-A_{k-1}=H_{k+1}$.

Since $\sigma_{k+2}, \sigma_{k+3}, \dots$ all vanish, we find for $j=k$ that

$$-A_k=\sum_{t=0}^{k+2}(-1)^{k-t}\binom{1}{t-k-1}H_tH_1'^{\,k+2-t},$$

whence

3°. $-A_k=-H_{k+1}H'_1+H_{k+2}$.

Similarly,

4°. $-A_{k+1}=H_{k+1}H'^2-2H_{k+2}H'+H_{k+3}$.

From 1° and 2° we again obtain (73), or

5°. $m_{n+k,\,n;\,0}=H_{k+1}$.

From 1° and 3° we find that $m_{n+k,\,n;\,1}=-\sigma_1 H_{k+1}+kH'_1 H_{k+1}-kH_{k+2}$. Since $-\sigma_1+kH'_1=-(H'_1+l_n+\ldots\ldots+l_{n+k})=-H_1$, we verify (74), or

6°. $m_{n+k,\,n;\,1}=-\{kH_{k+2}+H_1 H_{k+1}\}$.

From 1° and 4° we find that

$$m_{n+k,\,n;\,2}=\binom{k+1}{2}H_{k+3}+\binom{k+1}{1}H_1 H_{k+2}+H_{k+1}\{\sigma_1(H_1-H'_1)-kH'_1\tau_1-\tau_2 \\ -H'^2_1+2H'_1\bar{\bar{H}}'_1-\bar{\bar{H}}'_2\}.$$

In the coefficient of H_{k+1} the terms in k vanish, due to $\sigma_1=H_1+kH'_1$ and $H_1=\tau_1+\bar{\bar{H}}'_1=\tau_1+H'_1$; the coefficient then reduces to $H_1(H_1-H'_1)-\tau_2+\bar{\bar{H}}'^2_1-\bar{\bar{H}}'_2$. This becomes $H^2_1-H_2$, due to $H_2=\tau_2+\tau_1\bar{\bar{H}}'_1+\bar{\bar{H}}'_2$ and $\bar{\bar{H}}'_1=H'_1$. Hence,

7°. $m_{n+k,\,n;\,2}=\binom{k+1}{2}H_{k+3}+\binom{k+1}{1}H_1 H_{k+2}+(H^2_1-H_2)H_{k+1}$.

(79) *The third index number of $M_{n+k,\,n}$ is given in 7°.*

The method used to determine the first two index numbers in (73) and (74) failed for the third, because for the composite two-way there was required the second relative index number of $M_{n-k-1,\,n-1}$ as to $M_{n-k-1,\,n}$ (the $(\alpha\gamma)_1$ of (51)). This can now be found by using $m_{n+k,\,n;\,2}$, and as the result of a calculation similar to those made above we have

(80) *The second relative index number of $M_{n+k-1,\,n-1}$ as to $M_{n+k-1,\,n}$ is* $-H^{1,1}_{k+2}+\lambda_1 H^{1,1}_{k+1}$.

7. The index numbers of $M_{n+k,\,n}$ found thus far can be written

$$m_{n+k,\,n;\,0}=\left\{\binom{k-1}{0}H_{k+1}\right\},$$

$$m_{n+k,\,n;\,1}=-\left\{\binom{k}{1}H_{k+2}+\binom{k}{0}\bar{\bar{H}}_1 H_{k+1}\right\},$$

$$m_{n+k,\,n;\,2}=\left\{\binom{k+1}{2}H_{k+3}+\binom{k+1}{1}\bar{\bar{H}}_1 H_{k+2}+\binom{k+1}{0}\bar{\bar{H}}_2 H_{k+1}\right\}.$$

On this somewhat slender basis let us generalize the formulæ and assume as a tentative formula for the general index number of $M_{n+k,\,n}$

$$(81)\quad m_{n+k,\,n;\,j}=(-1)^j\left\{\binom{k+j-1}{j}H_{k+j+1}+\binom{k+j-1}{j-1}\bar{\bar{H}}_1 H_{k+j} \right. \\ +\binom{k+j-1}{j-2}\bar{\bar{H}}_2 H_{k+j-1}+\ldots\ldots \\ \left. +\binom{k+j-1}{1}\bar{\bar{H}}_{j-1} H_{k+2}+\binom{k+j-1}{0}\bar{\bar{H}}_j H_{k+1}\right\}.$$

In order to check this assumed formula, consider the matrix $M_{1+k,1}$. In this case

$$H_i=\mu_i+\lambda_1\mu_{i-1}+\lambda_i^2\mu_{i-2}+\ldots\ldots+\lambda_1^i\,;\quad i.\ e.,\quad H_i-\lambda_1 H_{i-1}=\mu_i.$$

Since $\mu_{k+2}=\mu_{k+3}=\ldots\ldots=0$,

$$H_{k+1}=(l_1+\lambda_1)\cdot\ldots\ldots\cdot(l_{1+k}+\lambda_1),\quad H_{k+1+r}=\lambda_1^r H_{k+1}.$$

The $M_{1+k,1}=0$ is regular, and its index numbers are formed from the complete symmetric functions of the orders; *i. e.*,

$$1^\circ.\quad m_{1+k,1;\,j}=(-1)^j H_{k+1}\left\{\bar\mu_j+\binom{k+j}{1}\bar\mu_{j-1}\lambda_1+\binom{k+j}{2}\bar\mu_{j-2}\lambda_1^2+\ldots\ldots+\binom{k+j}{j}\lambda_1^j\right\}.$$

The values of $\bar\mu_i$ in terms of the H's are

$$2^\circ.\qquad \bar\mu_j=\bar{\bar H}_j-\lambda_1\bar{\bar H}_{j-1}+\lambda_1^2\bar{\bar H}_{j-2}-\ldots\ldots+(-1)^j\lambda_1^j.$$

To prove this, we note that it is true for $j=1$; let us assume it true up to $j=j$, and prove it for $j=j$. From the assumed formula, $\bar\mu_i+\lambda_1\bar\mu_{i-1}=\bar{\bar H}_i$ for $i=1, 2, \ldots\ldots, j-1$. Since

$$\begin{aligned}\bar\mu_j&=\mu_1\bar\mu_{j-1}-\mu_2\bar\mu_{j-2}+\mu_3\bar\mu_{j-3}-\ldots\ldots+(-1)^j\mu_j\\&=(H_1-\lambda_1)\bar\mu_{j-1}-(H_2-\lambda_1 H_1)\bar\mu_{j-2}+(H_3-\lambda_1 H_2)\bar\mu_{j-3}\\&\qquad\qquad-\ldots\ldots+(-1)^{j-1}(H_j-\lambda_1 H_{j-1}),\\ \bar\mu_j+\lambda_1\bar\mu_{j-1}&=H_1(\bar\mu_{j-1}+\lambda_1\bar\mu_{j-2})-H_2(\bar\mu_{j-2}+\lambda_1\bar\mu_{j-3})+\ldots\ldots+(-1)^j H_j\\&=H_1\bar{\bar H}_{j-1}-H_2\bar{\bar H}_{j-2}+\ldots\ldots+(-1)^{j-1}H_j=\bar{\bar H}_j.\end{aligned}$$

This proves the above formula for $i=j$, and therefore completes the proof of 2°. Substituting the values 2° in 1°, we get

$$\begin{aligned}m_{1+k,1;\,j}&=(-1)^j\sum_{i=0}^{j}H_{k+1}\left[\binom{k+j}{i}-\binom{k+j}{i-1}+\binom{k+j}{i-2}-\ldots\ldots+(-1)^i\binom{k+j}{0}\right]\bar{\bar H}_{j-1}\lambda_1^i\\&=(-1)^j\sum_{i=0}^{j}\binom{k+j-1}{i}\bar{\bar H}_{j-i}\cdot\lambda_1^i H_{k+1}=(-1)^j\sum_{i=0}^{j}\binom{k+j-1}{i}\bar{\bar H}_{j-i}\cdot H_{k+1+i},\end{aligned}$$

which is the same result as is given by (81). In this particular case the H's, up to and including H_{k+1}, are independent quantities. The further H's are connected with the earlier ones by the equations $H_{k+1+i}=\lambda_1^i H_{k+1}$. The first homogeneous relation which is a consequence of these equations is $H_{k+1}H_{k+3}-H_{k+2}^2$, which can occur first in $m_{n+k,n;\,k+3}$.

(82) *On the assumption that the index numbers of $M_{n+k,n}$ can be given as polynomials in H_i, the formula* (81) *is correct for the first $k+3$ index numbers, and for these only except in the above case of an $M_{1+k,1}$.*

That this assumption is correct can hardly be doubted in view of the above values for the first three index numbers of $M_{n+k,\,n}$, and for all the index numbers of $M_{1+k,\,1}$, $M_{n-1,\,n}$ and $M_{n,\,n}$. That (81) is *not* correct for values $j \gtrless k+3$ is clear from the cases $M_{n,\,n}$ and $M_{n+1,\,n}$. The index numbers of the determinant $M_{n,\,n}$, according to [(11), "R. S.," I], are $m_{n,\,n;\,j}=(-1)^j H_1^j$, so that there must be added to the right side of (81) the following corrections for successive values of j: $0, 0, 0, 2(H_1H_3-H_2^2), -5H_1(H_1H_3-H_2^2), 9H_1^2(H_1H_3-H_2^2)+2(H_1^2H_4-3H_1H_2H_3+2H_2^3)+2(H_1H_5-4H_2H_4+3H_3^2), -14H_1^3(H_1H_3-H_2^2)-7H_1(H_1^2H_4-3H_1H_2H_3+2H_2^3)-7H_1(H_1H_5-4H_2H_4+3H_3^2), \ldots.$ In the case of an $M_{n+1,\,n}$ we find from (76) that the following corrections must be added to the right side of (81): $0, 0, 0, 0, 5(H_2H_4-H_3^2), \ldots.$

The only apparent law followed by these corrections is that they are seminvariants of the binary form with coefficients H_{k+1}, H_{k+2}, H_{k+3}, H_{k+4}, etc.

Let us make, finally, an application of the index numbers of $M_{n+k,\,n}$ to a geometric enumeration.

In S_{n+k-1} an S_{n-1} is determined by nk conditions, and it is one condition that an S_{n-1} meet a given S_{k-1} in S_{n+k-1}. We ask, then, for the number of S_{n-1}'s which meet nk given S_{k-1}'s in S_{n+k-1}. The S_{n-1} is given by n linearly independent points within it; *i. e.*, by

$$M_{n+k,\,n}=\left\|\begin{matrix} x_{1,\,1} & x_{1,\,2} & \ldots\ldots & x_{1,\,n+k} \\ x_{2,\,1} & x_{2,\,2} & \ldots\ldots & x_{2,\,n+k} \\ \ldots & \ldots & \ldots & \ldots \\ x_{n,\,1} & x_{n,\,2} & \ldots\ldots & x_{n,\,n+k} \end{matrix}\right\|.$$

If the variables x_{ij} be point coordinates in a space $\Sigma_{n(n+k)-1}$, the n points determine a point in $\Sigma_{n(n+k)-1}$. Since any other n linearly independent points will serve the same purpose, the S_{n-1} itself is represented by a Σ_{n^2-1} in $\Sigma_{n(n+k)-1}$. Take a section of $\Sigma_{n(n+k)-1}$ by a Σ_{nk}, and in Σ_{nk} the S_{n-1} is represented by a point. Conversely, a point in Σ_{nk} is given by values x_{ij} and determines an S_{n-1} in S_{n+k-1} unless the point of Σ_{nk} is on the spread defined by the vanishing of $M_{n+k,\,n}$. The condition that S_{n-1} meet a given S_{k-1} is linear in the determinants of $M_{n+k,\,n}$. It is therefore represented by a spread of order n in Σ_{nk} on the manifold $M_{n+k,\,n}=0$, whose dimension is $k(n-1)-1$. The number required is the number of points of Σ_{nk} outside of $M_{n+k,\,n}=0$, and on nk given spreads of order n containing $M_{n+k,\,n}=0$. This number is

$$(83)\qquad O=n^{nk}-\sum_{j=0}^{k(n-1)-1}\binom{nk}{nk-n-1-j}n^{nk-n-1-j}\,m_{n+k,\,n;\,j}\,.$$

(84) *The number of S_{n-1}'s which meet nk given S_{k-1}'s in S_{n+k-1} is given by O in (83), where $m_{n+k,\,n;\,j}$ is the $(j+1)$-th index number of a matrix $M_{n+k,\,n}$ whose elements are linear forms.*

9. Since, according to (82), we are limited to values $j<k+3$, *i. e.*, $k(n-1)-1<k+3$, we can obtain the explicit number only for $n=2$, $k=$ any integer, and for $n=3$, $k<4$. For a matrix $M_{n+k,\,n}$ with linear elements,

$$H_i=\binom{n+k}{i}, \quad \text{and} \quad \bar{H}_i=\binom{n+k-1+i}{i}.$$

Putting these values for $n=2$ in (81) and substituting in (83), we find that

$$\begin{aligned}O=2^{2k}-(k+2)\Big\{&\binom{k+1}{0}\binom{2k}{k-1}2^{k-1}-\binom{k+2}{1}\binom{2k}{k-2}2^{k-2}\\ &+\dots+(-1)^k\binom{2k}{k-1}\binom{2k}{0}2^0\Big\}\\ +\Big\{&\binom{k}{1}\binom{k+1}{0}\binom{2k}{k-2}2^{k-2}-\binom{k+1}{1}\binom{k+2}{1}\binom{2k}{k-3}2^{k-3}\\ &+\binom{k+2}{1}\binom{k+3}{2}\binom{2k}{k-4}2^{k-4}-\dots+(-1)^{k-1}\binom{2k-2}{1}\binom{2k-1}{k-2}\binom{2k}{0}2^0\Big\}.\end{aligned}$$

In the first brace note that

$$\binom{2k}{k-r}\binom{k+r}{r-1}=\binom{2k}{k-1}\binom{k-1}{r-1},$$

whence it becomes

$$\begin{aligned}-(k+2)\binom{2k}{k-1}\Big\{&\binom{k-1}{0}2^{k-1}-\binom{k-1}{1}2^{k-2}+\dots+(-1)^k\binom{k-1}{k-1}2^0\Big\}\\ &=-(k+2)\binom{2k}{k-1}(2-1)^{k-1}=-(k+2)\binom{2k}{k-1}.\end{aligned}$$

The second brace is

$$\begin{aligned}\sum_{r=0}^{k-2}(-1)^r\binom{k+r}{1}\binom{k+1+r}{r}\binom{2k}{k-2-r}2^{k-2-r}&=k\sum_{r=0}^{k-2}(-1)^r\binom{k+1+r}{r}\binom{2k}{k-2-r}2^{k-2-r}\\ &+(k+2)\sum_{r=1}^{k-2}(-1)^r\binom{k+1+r}{r-1}\binom{2k}{k-2-r}2^{k-2-r}.\end{aligned}$$

The first part of this, according to [11°, p. 179, "R. S.," I], is $k\sum_{r=2}^{k}\binom{2k}{k-r}$, and the second is $-(k+2)\sum_{r=3}^{k}\binom{2k}{k-r}$, whence the sum of the two is

$$k\binom{2k}{k-2}-2\sum_{r=3}^{k}\binom{2k}{k-r}=k\binom{2k}{k-2}-2^{2k}+2\binom{2k}{k-2}+2\binom{2k}{k-1}+\binom{2k}{k}.$$

Hence,

$$O=\binom{2k}{k}-k\binom{2k}{k-1}+(k+2)\binom{2k}{k-2}=\frac{(2k)\,!}{k\,!\,(k+1)\,!}=\frac{1}{k+1}\binom{2k}{k}.$$

(85) *The number of lines which meet $2k$ given S_{k-1}'s in S_{k+1} is*

$$\frac{(2k)\,!}{k\,!\,(k+1)\,!}=\binom{2k}{k}\frac{1}{k+1}.$$

This well-known fact can be regarded as an excellent numerical check on the previous theorems. The case $n=3$, $k=2$ of (84) is the dual of the case $n=2$, $k=3$ of (85), so that the case $n=3$, $k=3$ of (84) remains. From (81) we find that the index numbers of an $M_{6,3}$ in S_9 with linear elements are

$$15,\ -108,\ 465,\ -30\cdot 51,\ 21\cdot 210,\ -42\cdot 244,$$

whence

$$O=3^9-\binom{9}{5}3^5\cdot 15+\binom{9}{4}3^4\cdot 108-\binom{9}{3}3^3\cdot 465 +\binom{9}{2}3^2\cdot 30\cdot 51-\binom{9}{1}3\cdot 21\cdot 210+42\cdot 244=42.$$

(86) *There are forty-two planes which meet nine given planes in* S_5.

This again is checked by Schubert's formula,

$$\frac{1!2!3!\ldots r![(n-r)(r+1)]!}{(n-r)!(n-r+1)!\ldots(n-1)!\,n!},$$

for the number of S_r's which meet $(r+1)(n-r)$ given S_{n-r-1}'s in S .

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